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On Blaschke products associated with n-widths

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Abstract

Let E be a closed subset of the open unit disk $G = \{z : |z| < 1\}$, and let μ be a positive Borel measure with support supp $\mu = E$. Denote by $\mathbf{A}_p$ the restriction on E of the closed unit ball of the Hardy space $H_p(G)$, $1 \leqslant p \leqslant \infty$. In this paper we investigate orthogonality properties of the extremal functions associated with the Kolmogorov, Gelfand, and linear n-widths of $\mathbf{A}_p$ in $L_q(\mu, E)$, $1 \leqslant q < \infty$, $q \leqslant p$.

Keywords: n-widths; Blaschke product; Orthogonal polynomial; Meromorphic approximation

1. Introduction

1.1. Overview

Let $G = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbf{C}$ and let $\Gamma = \{z : |z| = 1\}$. We assume that the circle Γ is positively oriented with respect to G. Let E be a compact subset of G, and let μ be a finite positive Borel measure with support supp $\mu = E$. We further assume that E contains infinitely many points.

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Denote by $H_p(G)$, $1 \leqslant p \leqslant \infty$, the Hardy space of those analytic functions g on G such that $|g|^p$ has a harmonic majorant there. As is well known, such functions have nontangential boundary values a.e. on Γ that establish a one-to-one correspondence between $H_p(G)$ and the closed subspace of $L_p(\Gamma)$ consisting of functions whose Fourier coefficients of strictly negative index do vanish; a function in $H_p(G)$ is recovered from its boundary values through a Cauchy as well as a Poisson integral. We refer the reader to [9] for details on Hardy spaces, and we merely recall here a few facts that will be of relevance to us.

By a theorem of Szegő, we have that $\log |g| \in L_1(\Gamma)$ whenever $g \in H_p(G)$ is not the zero function. This entails that a $H_p(G)$-function is uniquely defined by the values it assumes on a subset of Γ of positive Lebesgue measure. Conversely, whenever $\rho \in L_p(\Gamma)$ is a positive function such that $\log \rho \in L_1(\Gamma)$, the function

$$E_\rho(z) = \exp\left\{\frac{1}{2\pi}\int_\Gamma \frac{\xi + z}{\xi - z}\log \rho(\xi)\, d|\xi|\right\}, \quad z \in G, \tag{1.1}$$

lies in $H_p(G)$ and has modulus ρ a.e. on Γ. This E_ρ is called the normalized *outer function* associated with ρ, the normalization being that $E_\rho(0) > 0$. More generally, a function is said to be *outer* in $H_p(G)$ if it is of the form cE_ρ with c a unimodular constant. Obviously, an outer function has no zero in G. Granted the normalization condition, the outer function E_ρ is characterized by two facts, namely:

(i) $|E_\rho| = \rho$ a.e. on Γ,
(ii) among all $H_p(G)$-functions that satisfy (i), E_ρ is largest-in-modulus pointwise on G.

A particular type of $H_\infty(G)$-functions will also be important to us, namely *finite Blaschke products*. These are the rational functions that are analytic in G and of unit modulus on Γ; they assume the form q/q^*, where q is an algebraic polynomial whose roots lie in G and where q^* indicates the *reciprocal polynomial* given by $q^*(z) = z^n\overline{q(1/\bar{z})}$ if n is the degree of q. The integer n is also called the degree of the Blaschke product, and the latter is called *normalized* if q is monic. For any positive integer n, we let $\mathscr{B}_n$ denote the class of normalized Blaschke products of degree n; upon splitting q into linear factors in the previous definition, we see that each $B \in \mathscr{B}_n$ can be uniquely written as

$$B(z) = \prod_{k=1}^{n} \frac{z - \xi_k}{1 - \bar{\xi}_k z}, \quad \xi_k \in G. \tag{1.2}$$

Let $\mathbf{A}_p$ be the restriction on E of the closed unit ball of the Hardy space $H_p(G)$. Fisher and Stessin [7,8] proved that in two important cases: when $1 \leqslant q \leqslant p < \infty$, or when $1 \leqslant q < \infty$, $p = 2$,

$$\begin{aligned} d^n(\mathbf{A}_p, L_q(\mu, E)) &= d_n(\mathbf{A}_p, L_q(\mu, E)) = \delta_n(\mathbf{A}_p, L_q(\mu, E)) \\ &= \inf_{B \in \mathscr{B}_n} \sup_{\varphi \in \mathbf{A}_p} ||\varphi B||_{q,\mu}, \end{aligned}$$

where d^n, d_n, and δ_n are the Kolmogorov, Gelfand and linear n-widths of $\mathbf{A}_p$ in the space $L_q(\mu, E)$ (see, for example, [10]), and $||\cdot||_{q,\mu}$ is the norm in the space $L_q(\mu, E)$.

Let $1 \leqslant q < \infty$, $1 \leqslant p \leqslant \infty$. Set

$$m_n = m_n(p, q, \mu) = \inf_{B \in \mathscr{B}_n} \sup_{\varphi \in \mathbf{A}_p} ||\varphi B||_{q,\mu}. \tag{1.3}$$

In this paper we investigate orthogonal properties of the extremal functions φ_n and B_n which attain the value m_n:

$$m_n = ||\varphi_n B_n||_{q,\mu}. \tag{1.4}$$

That φ_n and B_n indeed exist follows from the fact that the "inf-sup" in (1.3) is certainly attained if the infimization is extended to Blaschke product of degree *at most n*, because the restriction of this set to E is compact in $L_\infty(E)$ and so is $\mathbf{A}_p$ in $L_q(\mu, E)$; but the "inf" is obviously attained on $\mathscr{B}_n$, because each elementary factor in (1.2) has modulus strictly less than 1 on E. Necessarily φ_n is outer of $L_p(\Gamma)$-norm exactly 1, otherwise it could not meet the "sup" in (1.3). Clearly $\varphi_n \equiv 1$ for $p = \infty$, and for $1 \leqslant p < \infty$ is known to satisfy the following equation:

$$m_n^q \,|\, \varphi_n(\xi)|^p = \frac{1}{2\pi} \int_E \frac{1 - |x|^2}{|\xi - x|^2} |(\varphi_n B_n)(x)|^q \, d\mu(x), \quad \xi \in \Gamma, \tag{1.5}$$

see [7, Proposition 1]. One consequence is that $|\varphi_n|$ extends continuously on Γ (see [6,7]). Actually, since $|\varphi_n|$ is strictly positive and C^1-smooth on Γ by (1.5), so is $\log|\varphi_n|$ whose conjugate function Arg φ_n is then continuous, and therefore we see upon taking the exponential that the outer function φ_n itself is continuous on Γ. If $q \leqslant p$ then φ_n is uniquely determined by B_n up to unimodular scalar multiples, but this may fail if $p < q$ (it is nevertheless true if E is hyperbolically small, see [7]). To avoid trivial cases of nonuniqueness, we assume throughout without loss of generality that $\varphi_n(0) > 0$.

In Sections 2 and 3 we establish orthogonality properties of the extremal functions φ_n and B_n when $q \leqslant p$. The authors do not know whether analogous results hold when $p < q$. In Section 4, a connection with meromorphic approximation is investigated.

1.2. Notation

Above and thereafter, $L_p(\Gamma)$, $1 \leqslant p \leqslant \infty$, stands for the Lebesgue space of functions φ measurable on Γ, with the norm

$$||\varphi||_p = \left(\int_\Gamma |\varphi(\xi)|^p \, |d\xi| \right)^{1/p} \quad \text{if } 1 \leqslant p < \infty \tag{1.6}$$

and

$$||\varphi||_\infty = \operatorname*{ess\,sup}_\Gamma |\varphi(\xi)| \quad \text{if } p = \infty.$$

As well, $L_q(\mu, E)$, $1 \leqslant q < \infty$, is the Lebesgue space of functions φ on E with the norm

$$||\varphi||_{q,\mu} = \left(\int_E |\varphi(x)|^q \, d\mu(x) \right)^{1/q} < \infty.$$

Finally, for σ a positive Borel measure with support $\operatorname{supp} \sigma = E$, we denote by $J : H_2(G) \to L_2(\sigma, E)$ the *embedding* operator. The operator J is given by restricting an element $\varphi \in H_2(G)$ to E: $J\varphi = \varphi_{|E}$. Let $J^* : L_2(\sigma, E) \to H_2(G)$ be the adjoint of J. It is easily verified that for $\varphi \in H_2(G)$

$$(J^*J)(\varphi)(z) = \frac{1}{2\pi} \int_E \frac{\varphi(x)}{1 - z\bar{x}} \, d\sigma(x), \quad |z| < 1 \tag{1.7}$$

(see, for example, [5]).

2. Orthogonality properties of φ_n and B_n

Fix $1 \leqslant q < \infty$, $1 \leqslant p < \infty$, and a positive integer n. It is not hard to see that (1.5) is equivalent to the following relation:

$$\int_E u(x) \, |(\varphi_n B_n)(x)|^q \, d\mu(x) = m_n^q \int_\Gamma u(\xi) |\varphi_n(\xi)|^p |d\xi|, \tag{2.1}$$

where u is any function harmonic on G and continuous on $\bar{G}$. Equality (2.1) implies that

$$(|(\varphi_n B_n)(x)|^q \, d\mu(x))^*(\xi) = m_n^q |\varphi_n(\xi)|^p |d\xi|, \quad \xi \in \Gamma,$$

where $(|\varphi_n B_n|^q d\mu)^*$ is the balayage of $|\varphi_n B_n|^q \, d\mu$ on Γ (see, for example, [11]). It follows from (2.1) that for any $g \in H_1(G)$

$$\int_E g(x) |(\varphi_n B_n)(x)|^q \, d\mu(x) = m_n^q \int_\Gamma g(\xi) |\varphi_n(\xi)|^p |d\xi|. \tag{2.2}$$

We represent $B_n(x)$ in the form $B_n(x) = w_n(x)/w_n^*(x)$, where

$$w_n(x) = \prod_{k=1}^n (x - x_{k,n}), \quad w_n^*(x) = \prod_{k=1}^n (1 - \bar{x}_{k,n} x),$$

and $x_{1,n}, x_{2,n}, \ldots, x_{n,n}$ are zeros of B_n, $x_{k,n} \in G$, $k = 1, \ldots, n$.

We now prove that for $1 \leqslant q \leqslant p < \infty$

$$\int_E \left(\frac{x^k}{w_n^*(x)} \overline{B_n(x)} - \overline{\left(\frac{x^{n-k} w_n(x)}{(w_n^*(x))^2} \right)} B_n(x) \right) |\varphi_n(x)|^q |B_n(x)|^{q-2} \, d\mu(x) = 0,$$
$$k = 0, \ldots, n-1. \tag{2.3}$$

For $B \in \mathscr{B}_n$ and $\varphi \in \mathbf{A}_p$, we set

$$\Psi(B, \varphi) \triangleq \int_E |(\varphi B)(x)|^q \, d\mu(x)$$

and we denote by φ_B the unique function (up to unimodular scalar multiples) in $\mathbf{A}_p$ such that

$$\Psi(B, \varphi_B) = \sup_{\varphi \in \mathbf{A}_p} \Psi(B, \varphi). \tag{2.4}$$

Necessarily φ_B is outer, so we normalize it as usual by setting $\varphi_B(0) > 0$. We write a generic $B \in \mathscr{B}_n$ as:

$$B(x) = \frac{\alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} + x^n}{\bar{\alpha}_0 x^n + \bar{\alpha}_1 x^{n-1} + \cdots + \bar{\alpha}_{n-1} x + 1}, \quad \alpha_j \in \mathbf{C},$$

and we single out B_n to be

$$B_n(x) = \frac{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n}{\bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \cdots + \bar{a}_{n-1} x + 1},$$

where the a_j are the coefficients of w_n.

When B ranges over $\mathscr{B}_n$, then $(\alpha_0, \ldots, \alpha_{n-1})$ ranges over an open subset Ω of $\mathbf{C}^n$, and this way we coordinatize $\mathscr{B}_n$. Clearly, Ψ is jointly continuous with respect to $(\alpha_j)_{0 \leqslant j \leqslant n-1} \in \Omega$ and $\varphi \in \mathbf{A}_p$ when the latter is endowed with the topology induced by the sup-norm on E. Observe also that $\mathbf{A}_p$ is compact for that topology by the Cauchy formula and the weak-compactness of balls in $H_p(G)$. Moreover, Ψ has partial derivatives with respect to the real and imaginary parts of the α_j that are likewise jointly continuous with respect to $(\alpha_j)_{0 \leqslant j \leqslant n-1}$ and φ. Since φ_B is the unique argument of the maximum in (2.4), it now follows from [4, Chapter III, Theorem 1] that $\Psi(B, \varphi_B)$ in turn has partial derivatives with respect to the real and imaginary parts of the α_j, given by

$$\frac{\partial \Psi(B, \varphi_B)}{\partial \mathrm{Re}(\alpha_j)} = \left.\frac{\partial \Psi(B, \varphi)}{\partial \mathrm{Re}(\alpha_j)}\right|_{\varphi = \varphi_B}, \quad \frac{\partial \Psi(B, \varphi_B)}{\partial \mathrm{Re}(\alpha_j)} = \left.\frac{\partial \Psi(B, \varphi)}{\partial \mathrm{Re}(\alpha_j)}\right|_{\varphi = \varphi_B}.$$

Because $B \mapsto \Psi(B, \varphi_B)$ meets a minimum on $\mathscr{B}_n$ at $B = B_n$, or in coordinates on Ω at $\alpha_j = a_j$, $0 \leqslant j \leqslant n-1$, the above partial derivatives must vanish at this point and writing that

$$\left.\left(\frac{\partial \Psi(B, \varphi_B)}{\partial \mathrm{Re}(\alpha_k)} - i \frac{\partial \Psi(B, \varphi_B)}{\partial \mathrm{Im}(\alpha_k)}\right)\right|_{B = B_n} = 0,$$

while taking into account that $\varphi_{B_n} = \varphi_n$, yields (2.3) upon differentiating under the integral sign.

Let E^{-1} be the reflection of E in the unit circle and let D be the component of $\bar{\mathbf{C}} \backslash E^{-1}$ that contains the unit disk G. We have the following theorem (see [2] for $q = 2$ and $E \subset (-1, 1)$).

Theorem 1. *The function* $\varphi_n^{p/2}$ *can be extended analytically to* D, *and satisfies the equations*

$$m_n^q(\varphi_n^{p/2})(\xi) = \frac{1}{2\pi}\int_E \frac{|(\varphi_n B_n)(x)|^q}{(1-\xi\bar{x})\overline{\varphi_n^{p/2}(x)}}\,d\mu(x), \quad \xi \in D, \quad 1 \leqslant q < \infty, 1 \leqslant p < \infty \tag{2.5}$$

and

$$m_n^q(B_n\varphi_n^{p/2})(\xi) = \frac{1}{2\pi}\int_E \frac{|(\varphi_n B_n)(x)|^q}{(1-\xi\bar{x})\overline{B_n(x)\varphi_n^{p/2}(x)}}\,d\mu(x), \quad \xi \in D, 1 \leqslant q \leqslant p < \infty. \tag{2.6}$$

The following orthogonality relations are valid:

$$\int_E \frac{x^k\overline{w_n(x)}}{|w_n^*(x)|^2\varphi_n^{p/2}(x)}|\varphi_n(x)|^q|B_n(x)|^{q-2}\,d\mu(x) = 0, \; k = 0, \dots, n-1,$$
$$1 \leqslant q \leqslant p < \infty. \tag{2.7}$$

Proof. Let φ be any function in $H_2(G)$, and let $g = \varphi/\varphi_n^{p/2}$. Here and in what follows we take that branch of the $(p/2)$th root that is positive on the positive part of the real line. By (2.2), we can write

$$\int_E \varphi(x)\overline{\varphi_n^{p/2}(x)}|\varphi_n(x)|^{q-p}\,|\,B_n(x)|^q\,d\mu(x) = m_n^q\int_\Gamma \varphi(\xi)\overline{\varphi_n^{p/2}(\xi)}|d\xi|.$$

Therefore,

$$J^*J(\varphi_n^{p/2}) = m_n^q\varphi_n^{p/2},$$

where $J : H_2(G) \to L_2(|\varphi_n|^{q-p}|B_n|^q d\mu, E)$ is the embedding operator. From this, on the basis of formula (1.7) where $d\sigma = |\varphi_n|^{q-p}|B_n|^q\,d\mu$, we obtain (2.5).

We now can rewrite (2.3) in the form

$$\int_E \frac{x^k}{w_n(x)}|(\varphi_n B_n)(x)|^q\,d\mu(x) = \int_E \left(\overline{\frac{x^{n-k}}{w_n^*(x)}}\right)|(\varphi_n B_n)(x)|^q\,d\mu(x).$$

By (2.2),

$$\int_E \left(\overline{\frac{x^{n-k}}{w_n^*(x)}}\right)|(\varphi_n B_n)(x)|^q\,d\mu(x) = m_n^q\int_\Gamma \left(\overline{\frac{\xi^{n-k}}{w_n^*(\xi)}}\right)|\varphi_n(\xi)|^p|d\xi|$$
$$= m_n^q\int_\Gamma \frac{\xi^k}{w_n(\xi)}|\varphi_n(\xi)|^p|d\xi|. \tag{2.8}$$

Hence, we have

$$\int_E \frac{x^k}{w_n(x)}|(\varphi_n B_n)(x)|^q\,d\mu(x) = m_n^q\int_\Gamma \frac{\xi^k}{w_n(\xi)}|\varphi_n(\xi)|^p|d\xi|. \tag{2.9}$$

By (2.2) and (2.9), for any $g \in H_1(G)$

$$\int_E \frac{g(x)}{w_n(x)} |(\varphi_n B_n)(x)|^q \, d\mu(x) = m_n^q \int_\Gamma \frac{g(\xi)}{w_n(\xi)} |\varphi_n(\xi)|^p |d\xi|. \tag{2.10}$$

Letting

$$g(x) = \frac{w_n^*(x)}{(1-\bar{t}x)\varphi_n^{p/2}(x)}, \quad |t|<1,$$

we get

$$\begin{aligned}
&\frac{1}{2\pi} \int_E \frac{|(\varphi_n B_n)(x)|^q}{(1-\bar{t}x)B_n(x)\varphi_n^{p/2}(x)} \, d\mu(x) \\
&\quad = \frac{m_n^q}{2\pi} \int_\Gamma \frac{\overline{B_n(\xi)\varphi_n^{p/2}(\xi)}|d\xi|}{(1-\bar{t}\xi)} = m_n^q \overline{\frac{1}{2\pi i} \int_\Gamma \frac{B_n(\xi)\varphi_n^{p/2}(\xi) d\xi}{\xi - t}} \\
&\quad = m_n^q \overline{B_n(t)\varphi_n^{p/2}(t)},
\end{aligned}$$

and, then, (2.6).

By (2.10), for $g = x^k/\varphi_n^{p/2}$, $k = 0, \dots, n-1$, we obtain that

$$\begin{aligned}
&\int_E \frac{x^k}{w_n(x)} \frac{|(\varphi_n B_n)(x)|^q}{\varphi_n^{p/2}(x)} \, d\mu(x) = m_n^q \int_\Gamma \frac{\xi^k \overline{\varphi_n^{p/2}(\xi)}|d\xi|}{w_n(\xi)} \\
&\quad = m_n^q \overline{\int_\Gamma \frac{\xi^{n-k}\varphi_n^{p/2}(\xi)|d\xi|}{w_n^*(\xi)}} = m_n^q \overline{\int_\Gamma \frac{\xi^{n-k-1}\varphi_n^{p/2}(\xi) \, d\xi}{i w_n^*(\xi)}};
\end{aligned}$$

consequently,

$$\int_E \frac{x^k}{w_n(x)} \frac{|(\varphi_n B_n)(x)|^q}{\varphi_n^{p/2}(x)} \, d\mu(x) = 0, \quad k = 0, \dots, n-1,$$

and (2.7) follows. □

3. The case $p = \infty$

Let $1 \leqslant q < \infty$, $p = \infty$, and let n be a positive integer. Since in this case

$$\sup_{\varphi \in A_\infty} ||\varphi B||_{q,\mu} = ||B||_{q,\mu}, \quad B \in \mathscr{B}_n,$$

we can rewrite (1.3) and (1.4) in the form

$$m_n = \inf_{B \in \mathscr{B}_n} ||B||_{q,\mu} = ||B_n||_{q,\mu}. \tag{3.1}$$

Let us consider the following function:

$$g_n(\xi) = \frac{1}{2\pi} \int_E \frac{1-|x|^2}{|\xi - x|^2} B_n(x)|^q \, d\mu(x), \quad \xi \in \Gamma. \tag{3.2}$$

Let u be any harmonic function on G that is continuous on the closed disk $\bar{G}$. By (3.2),

$$\int_E u(x)|B_n(x)|^q \, d\mu(x) = \int_\Gamma u(\xi) g_n(\xi) |d\xi|, \tag{3.3}$$

and consequently

$$(|B_n|^q \, d\mu)^*(\xi) = g_n(\xi)|d\xi|, \quad \xi \in \Gamma,$$

where $(|B_n|^q \, d\mu)^*$ is the balayage of $|B_n|^q \, d\mu$ on Γ. In particular, we have that

$$||g_n||_{L_1(\Gamma)} = m_n^q. \tag{3.4}$$

Consider now the function $\phi_n(z)$ defined on G by

$$\phi_n(z) = \exp\left(\frac{1}{4\pi} \int_\Gamma \frac{\xi + z}{\xi - z} \log |g_n(\xi)/m_n^q| d\xi|\right).$$

Because g_n is strictly positive and continuous on Γ as is apparent from (3.2), the function ϕ_n is normalized-outer in $H_\infty(G)$ by (1.1) and, from the properties of such functions (as described in the introduction), together with (3.4), ϕ_n satisfies the following three properties:

(a) ϕ_n is nonvanishing in G;
(b) $||\phi_n||_2 = 1$ and $\phi_n(0) > 0$;
(c) ϕ_n satisfies on Γ the equation

$$g_n(\xi) = m_n^q |\phi_n(\xi)|^2. \tag{3.5}$$

Moreover, since g_n is C^1-smooth and non-vanishing on Γ by (3.2), so is $|\phi_n|$ and therefore ϕ_n itself is continuous on Γ.

With the aid of ϕ_n we shall prove the following version of Theorem 1 for the case when $p = \infty$ (see [3] for $q = 2$ and $E \subset (-1, 1)$).

Theorem 2. *Let $p = \infty$ and $1 \leqslant q < \infty$. The function ϕ_n can be extended analytically to D, and satisfies the equations*

$$m_n^q \phi_n(\xi) = \frac{1}{2\pi} \int_E \frac{|B_n(x)|^q \, d\mu(x)}{(1 - \xi\bar{x})\overline{\phi_n(x)}}, \quad \xi \in D \tag{3.6}$$

and

$$m_n^q (\phi_n B_n)(\xi) = \frac{1}{2\pi} \int_E \frac{|B_n(x)|^q \, d\mu(x)}{(1 - \xi\bar{x})\overline{\phi_n(x) B_n(x)}}, \quad \xi \in D. \tag{3.7}$$

The following orthogonality relations are valid:

$$\int_E \frac{x^k \overline{w_n(x)}}{|w_n^*(x)|^2 \phi_n(x)} |B_n(x)|^{q-2} \, d\mu(x) = 0, \quad k = 0, \ldots, n-1. \tag{3.8}$$

Proof. Let g be any function in $H_1(G)$. Using (3.2), (3.3), and (3.5), we can write

$$\int_E g(x)|B_n(x)|^q \, d\mu(x) = m_n^q \int_\Gamma g(\xi)|\phi_n(\xi)|^2 |d\xi|. \tag{3.9}$$

It follows from the last formula that

$$J^* J(\phi_n) = m_n^q \phi_n, \tag{3.10}$$

where $J : H_2(G) \to L_2(|B_n|^q/|\phi_n|^2 \, d\mu, E)$ is the embedding operator. By (3.10) and (1.7), we get (3.6).

Since for any Blaschke product $B \in \mathscr{B}_n$

$$\int_E |B(x)|^q \, d\mu(x) \geqslant \int_E |B_n(x)|^q \, d\mu(x), \tag{3.11}$$

it follows that for $k = 0, \dots, n-1$,

$$\int_E \left(\frac{x^k}{w_n^*(x)} \overline{B_n}(x) - \left(\overline{\frac{x^{n-k} w_n(x)}{(w_n^*(x))^2}} \right) B_n(x) \right) |B_n(x)|^{q-2} \, d\mu(x) = 0. \tag{3.12}$$

We can see from (3.12) and (3.9) that

$$\begin{aligned} &\int_E \frac{x^k}{w_n(x)} |B_n(x)|^q \, d\mu(x) \\ &= \int_E \left(\overline{\frac{x^{n-k}}{w_n^*(x)}} \right) |B_n(x)|^q \, d\mu(x) = m_n^q \int_\Gamma \left(\overline{\frac{\xi^{n-k}}{w_n^*(\xi)}} \right) |\phi_n(\xi)|^2 |d\xi| \\ &= m_n^q \int_\Gamma \frac{\xi^k}{w_n(\xi)} |\phi_n(\xi)|^2 |d\xi|. \end{aligned} \tag{3.13}$$

By (3.9) and (3.13), for any $g \in H_1(G)$

$$\int_E \frac{g(x)}{w_n(x)} |B_n(x)|^q \, d\mu(x) = m_n^q \int_\Gamma \frac{g(\xi)}{w_n(\xi)} |\phi_n(\xi)|^2 |d\xi|. \tag{3.14}$$

Letting

$$g(x) = \tfrac{w_n^*(x)}{(1-\bar{t}x)\phi_n(x)}, \ |t| < 1,$$

we get

$$\frac{1}{2\pi} \int_E \frac{|B_n(x)|^q}{(1-\bar{t}x) B_n(x) \phi_n(x)} = \frac{m_n^q}{2\pi} \int_\Gamma \frac{\overline{B_n(\xi)\phi_n(\xi)} |d\xi|}{(1-\bar{t}\xi)},$$

$$m_n^q \overline{\frac{1}{2\pi i} \int_\Gamma \frac{B_n(\xi)\phi_n(\xi) \, d\xi}{\xi - t}} = m_n^q \overline{B_n(t)\phi_n(t)}$$

and, then, (3.7).

By (3.14), for

$$g = \frac{x^k}{\phi_n}, k = 0, \dots, n-1,$$

we obtain that

$$\int_E \frac{x^k}{w_n(x)} \frac{|B_n(x)|^q}{\phi_n(x)} d\mu(x) = m_n^q \int_\Gamma \frac{\xi^k \overline{\phi_n(\xi)} |d\xi|}{w_n(\xi)}$$
$$= m_n^q \overline{\int_\Gamma \frac{\xi^{n-k} \phi_n(\xi) |d\xi|}{w_n^*(\xi)}} = m_n^q \overline{\int_\Gamma \frac{\xi^{n-k-1} \phi_n(\xi)\, d\xi}{i w_n^*(\xi)}};$$

consequently,

$$\int_E \frac{x^k}{w_n(x)} \frac{|B_n(x)|^q}{\phi_n(x)} d\mu(x) = 0, \quad k = 0, \dots, n-1,$$

and (3.8) follows. □

4. Connection with meromorphic approximation

For $q = 2$ and $E \subset (-1, 1)$ there is an important connection between the best meromorphic approximation error of the Markov function

$$f(z) = \frac{1}{2\pi i} \int_E \frac{d\mu(x)}{z - x} \tag{4.1}$$

and the extremal constant m_n.

Let $\mathcal{M}_{n,s}(G)$, $1 \leqslant s \leqslant \infty$, be the class of all meromorphic functions on G that can be represented in the form $h = P/Q$, where P belongs to the Hardy space $H_s(G)$ and Q is a polynomial of degree at most n, $Q \not\equiv 0$. Denote by

$$\Delta_{n,s} = \inf_{h \in \mathcal{M}_{n,s}(G)} ||f - h||_s$$

the error in best approximation of the Markov function f in the space $L_s(\Gamma)$ by meromorphic functions in the class $\mathcal{M}_{n,s}(G)$.

Let $1/s + 1/t = 1$. The following theorem describes a connection between $\Delta_{n,s}$ and the extremal constant $m_n(p, q, \mu)$ with $q = 2$ and $p = 2t$ (see [1,2]).

Theorem 3. (i) *We have*

$$\Delta_{n,s} = m_n^2(2t, 2, \mu);$$

(ii) *there exists a best meromorphic approximant* h_n *in* $\mathcal{M}_{n,s}(G)$ *to the Markov function* f *in the space* $L_s(\Gamma)$ *such that*

$$\Delta_{n,s} = ||f - h_n||_s$$

and

$$h_n = P_n/B_n,$$

where $P_n \in H_s(G)$ *and* B_n *is a solution of the extremal problem* (1.3) *with* $q = 2$ *and* $p = 2t$;

(iii) *the function* h_n *satisfies a.e. on* Γ *the following equations*:

$$(\varphi_n^2 B_n^2)(\xi)(f - h_n)(\xi)\, d\xi = \Delta_{n,s}|\varphi_n(\xi)|^{2t}|\, d\xi| \quad \textit{if} \ \ 1 < s \leqslant \infty,$$

and

$$(B_n^2(f - h_n))(\xi)\, d\xi = |(f - h_n)(\xi)||d\xi| \quad \textit{if} \ \ s = 1.$$

Proof. We shall show that this theorem follows easily from results of Sections 2 and 3. Without loss of generality we assume that $1 < s \leqslant \infty$. Let

$$m_n = m_n(2t, 2, \mu) = \inf_{B \in \mathscr{B}_n} \sup_{\varphi \in \mathbf{A}_{2t}} ||\varphi B||_{2,\mu} = ||\varphi_n B_n||_{2,\mu}. \tag{4.2}$$

Since $E \subset \mathbf{R}$ it is not hard to prove that all zeros $x_{1,n}, \ldots, x_{n,n}$ of B_n belong to the smallest interval $K(E)$ containing support supp $\mu = E$ of μ (see, for example, [1]). Using (1.5) with $q = 2$ and $p = 2t$, we can write $|\varphi_n(\bar{\xi})| = |\varphi_n(\xi)|$ for $\xi \in \Gamma$. Since φ_n is outer, it follows from this that

$$\overline{\varphi_n(\bar{\xi})} = c\varphi_n(\xi), \quad \xi \in G, \quad |c| = 1. \tag{4.3}$$

As above we can assume that $\varphi_n(0) > 0$. Then (4.3) yields $\varphi_n > 0$ on $(-1, 1)$.

By (2.10), for any function $g \in H_1(G)$ we get

$$\int_E g(x)\varphi_n^2(x)B_n(x)\, d\mu(x) = m_n^2 \int_\Gamma g(\xi)\overline{B_n(\xi)}|\varphi_n(\xi)|^{2t}|d\xi|$$

and (see (4.1))

$$\int_\Gamma g(\xi)\varphi_n^2(\xi)B_n(\xi)f(\xi)\, d\xi = m_n^2 \int_\Gamma g(\xi)\overline{B_n(\xi)}|\varphi_n(\xi)|^{2t}|d\xi|. \tag{4.4}$$

Since (4.4) is valid for any $g \in H_1(G)$, it follows (see, for example, [9]) that there exists a function $p \in H_\infty(G)$ such that

$$\varphi_n^2(\xi)B_n(\xi)f(\xi) - m_n^2\overline{B_n(\xi)}|\varphi_n(\xi)|^{2t}|d\xi|/d\xi = p(\xi)$$

a.e. on Γ. From this we obtain that

$$\varphi_n^2(\xi)B_n^2(\xi)\left(f(\xi) - \frac{p(\xi)}{\varphi_n^2(\xi)B_n(\xi)}\right) d\xi = m_n^2|\varphi_n(\xi)|^{2t}|d\xi| \tag{4.5}$$

a.e. on Γ. Since $||\varphi_n||_{2t} = 1$, we can conclude from the last relation that

$$||f - h_n||_s = m_n^2, \tag{4.6}$$

where $h_n = P_n/B_n$ and $P_n = p/\varphi_n^2$. We remark that $h_n \in \mathscr{M}_{n,s}(G)$.

By the duality results (see [9]), (4.2) and (4.1), we get

$$\Delta_{n,s} = \inf_{B \in \mathscr{B}_n} \sup_{\varphi \in \mathbf{A}_t} \left| \int_\Gamma (\varphi B f)(\xi)\, d\xi \right|$$
$$= \inf_{B \in \mathscr{B}_n} \sup_{\varphi \in \mathbf{A}_t} \left| \int_E \varphi(x) B(x)\, d\mu(x) \right| \geqslant m_n^2. \tag{4.7}$$

Therefore, by (4.6), (4.7), and the fact that the function $h_n \in \mathscr{M}_{n,s}(G)$, we get

$$m_n^2 = \Delta_{n,s} = ||f - h_n||_s. \quad \square$$

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